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ORDINARY DIFFERENTIAL EQUATIONS

On the Dichotomy of a System of Linear Differential Equations with Conditionally Periodic Coefficients

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Abstract—We show that a system of linear differential equations with conditionally periodic coefficients is exponentially dichotomous if and only if the spectrum of the monodromy operator does not meet the unit circle.

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0. INTRODUCTION

Consider a system of linear differential equations with conditionally periodic coefficients,

$$x' = a(t)x, \tag{0.1}$$

where x is an n-vector, and $a(t)$ is a continuous conditionally periodic $n \times n$ matrix. The latter means [1] that there exists a continuous ω-periodic matrix $A(\varphi)$, $\varphi = (\varphi_1, \ldots, \varphi_m)$, $\omega = (\omega_1, \ldots, \omega_m)$, $m \geq 2$, such that the frequencies $\beta_i = 2\pi/\omega_i$ are rationally incommensurable and $a(t) = A(et)$ with the m-vector $e = (1, \ldots, 1)$.

In the present paper, we analyze the conditionally periodic system (0.1) using an approach in which this system is replaced by a system of partial differential equations [2] (or a system of integral equations [3]). The diagonal function $x(t) = u(et)$ of a solution $u = u(\varphi)$ of the new system is a solution of system (0.1). This permits one to define the monodromy operator [3] of system (0.1), which is an analog of the monodromy matrix of a periodic system.

Dichotomy problems for almost periodic and periodic systems of differential equations were considered in the monographs [4, 5]. The main result of the present paper, Theorem 2.1, is an analog of the classical dichotomy theorem for a periodic system [5, p. 288].

The monodromy operator belongs to the class of weighted shift operators considered in the monograph [6]. The results of this monograph imply that the spectrum of the monodromy operator is invariant under rotations around zero (Lemma 1.1 of the present paper).

Consider the matrix equation

$$X(\varphi) = \int\limits_0^{\varphi_1 - \varphi_{10}} A(\varphi - e\xi)X(\varphi - e\xi)\, d\xi + \Psi(\hat{\varphi} - \hat{e}(\varphi_1 - \varphi_{10})), \tag{0.2}$$

where $\Psi(\hat{\varphi})$ is a given $n \times n$ matrix, $X(\varphi)$ is the unknown $n \times n$ matrix, $\hat{\varphi} = (\varphi_2, \ldots, \varphi_m)$, and $\hat{e} = (1, \ldots, 1)$ is an $(m-1)$-vector. By $X(\varphi; \varphi_{10})$ we denote the solution of Eq. (0.2) with $\Psi(\hat{\varphi}) = E$ (the identity matrix) and $X_0(\varphi) = X(\varphi; 0)$.

The following assertion was proved in [3].

Lemma 0.1. *Let A be a continuous ω-periodic matrix function on R^m, and let Ψ be a continuous matrix function nonsingular for all $\hat{\varphi} \in R^{m-1}$; then Eq.* (0.2) *has a unique solution, which can be* *represented in the form* $X(\varphi) = X(\varphi;\varphi_{10})\Psi(\hat{\varphi} - \hat{e}(\varphi_1 - \varphi_{10}))$. *The matrices* $X(\varphi)$, $X(\varphi;\varphi_{10})$, *and* $X_0(\varphi)$ *have the following properties.*

(a) $X(\varphi)$, $X(\varphi;\varphi_{10})$, *and* $X_0(\varphi)$ *are continuous matrix functions nonsingular for all* $\varphi \in R^m$ *and* $\varphi_{10} \in R$.

(b) $X(\varphi; s) = X(\varphi;\tau)X(\varphi - e(\varphi_1 - \tau); s)$, $\varphi \in R^m$, $s, \tau \in R$.

(c) $X(\varphi;\varphi_{10}) = X(\varphi)X^{-1}(\varphi - e(\varphi_1 - \varphi_{10})) = X_0(\varphi)X_0^{-1}(\varphi - e(\varphi_1 - \varphi_{10}))$, *and* $X(\varphi_{10}, \hat{\varphi}; \varphi_{10}) = E$.

(d) $X(\varphi_1 + \omega_1, \hat{\varphi}; \varphi_{10}) = X(\varphi;\varphi_{10})X(\varphi_{10} + \omega_1, \hat{\varphi} - \hat{e}(\varphi_1 - \varphi_{10}); \varphi_{10})$; *in particular,*

$$X_0(\varphi_1 + \omega_1, \hat{\varphi}) = X_0(\varphi)X_0(\omega_1, \hat{\varphi} - \hat{e}\varphi_1).$$

[*Here the vector* φ *has the form* $(\varphi_1, \hat{\varphi})$.]

(e) *The matrices* $X(\varphi;\varphi_{10})$ *and* $X_0(\varphi)$ *are* $\hat{\omega}$*-periodic in* $\hat{\varphi}$ *and the matrix* X *is* $\hat{\omega}$*-periodic in* $\hat{\varphi}$ *if so is the matrix* Ψ, *where* $\hat{\omega} = (\omega_2, \ldots, \omega_m)$.

(f) *The relation*

$$\frac{dX(\varphi + et; \varphi_{10})}{dt} = A(\varphi + et)X(\varphi + et; \varphi_{10})$$

holds for arbitrary $\varphi \in R^m$, $\varphi_{10} \in R$, *and* $t \in R$; *here* $X_0(et)$ *is the normalized principal solution matrix of system* (0.1).

1. INHOMOGENEOUS SYSTEM

Consider the inhomogeneous system

$$x' = a(t)x + f(t), \tag{1.1}$$

where f is a continuous conditionally periodic vector function, $f(t) = F(et)$, $F \in P_n^0(\omega)$, $P_n^0(\omega)$ is the Banach space of continuous ω-periodic vector functions $F: R^m \to R^n$ with norm $\|F\| = \sup_{\varphi \in R^m} |F(\varphi)|$, and $|\cdot|$ is a norm on the finite-dimensional space. The space of conditionally periodic functions $f(t) = F(et)$, $t \in R$, $F \in P_n^0(\omega)$, is denoted by $B_n^0(\omega)$.

Theorem 1.1. *Let* $f \in B_n^0(\omega)$. *System* (1.1) *has a solution* $x \in B_n^0(\omega)$ *if and only if there exists a function* $u_0 \in P_n^0(\hat{\omega})$ *such that the function*

$$u(\varphi) = X_0(\varphi)u_0(\hat{\varphi} - \hat{e}\varphi_1) + \int_0^{\varphi_1} X(\varphi;\xi)F(\xi, \hat{\varphi} - \hat{e}\varphi_1 + \hat{e}\xi)\,d\xi \tag{1.2}$$

is an element of the space $P_n^0(\omega)$. *The solution has the form* $x(t) = u(et)$.

Proof. Let $u \in P_n^0(\omega)$. We replace φ in relation (1.2) by $\varphi + et$ and differentiate both sides of the resulting relation with respect to t,

$$\begin{aligned}\frac{du(\varphi + et)}{dt} &= A(\varphi + et)X_0(\varphi + et)u_0(\hat{\varphi} - \hat{e}\varphi_1) + X(\varphi + et; \varphi_1 + t)F(\varphi_1 + t, \hat{\varphi} + \hat{e}t)\\ &+ \int_0^{\varphi_1 + t} A(\varphi + et)X(\varphi + et;\xi)F(\xi, \hat{\varphi} - \hat{e}\varphi_1 + \hat{e}\xi)d\xi = A(\varphi + et)u(\varphi + et) + F(\varphi + et).\end{aligned}$$

Therefore, $u(et)$ is a conditionally periodic solution of system (1.1).

Conversely, let x be a conditionally periodic solution of system (1.1), $x(t) = u(et)$; then for each $\tau \in R$, we have

$$\frac{du(e(t + \tau))}{dt} = A(e(t + \tau))u(e(t + \tau)) + F(e(t + \tau)).$$

For each $\hat{\varphi} \in R^{m-1}$, there exists a sequence $\{\tau_n\}$ such that the sequences

$$\{u(e(t+\tau_n))\}, \qquad \{A(e(t+\tau_n))\}, \qquad \{F(e(t+\tau_n))\}$$

converge uniformly on R to $u(t, \hat{e}t + \hat{\varphi})$, $A(t, \hat{e}t + \hat{\varphi})$, and $F(t, \hat{e}t + \hat{\varphi})$, respectively, as $n \to +\infty$ [1, p. 340; 7, p. 126]; consequently,

$$\frac{du(t, \hat{e}t + \hat{\varphi})}{dt} = A(t, \hat{e}t + \hat{\varphi})u(t, \hat{e}t + \hat{\varphi}) + F(t, \hat{e}t + \hat{\varphi}). \qquad (1.1_{\hat{\varphi}})$$

The solution of this system can be represented in the form

$$u(t, \hat{e}t + \hat{\varphi}) = X_0(t, \hat{e}t + \hat{\varphi})u(0, \hat{\varphi}) + \int_0^t X(t, \hat{e}t + \hat{\varphi}; \xi)F(\xi, \hat{e}\xi + \hat{\varphi})\, d\xi.$$

By setting $\psi_1 = t$, $\hat{\psi} = \hat{\varphi} + \hat{e}t$, and $\psi = (\psi_1, \hat{\psi})$, we obtain

$$u(\psi) = X_0(\psi)u(0, \hat{\psi} - \hat{\psi}_1) + \int_0^{\psi_1} X(\psi; \xi)F(\xi, \hat{e}(\xi - \psi_1) + \hat{\psi})\, d\xi.$$

We have thereby obtained a function $u \in P_n^0(\omega)$ of the form (1.2) with $u_0(\hat{\varphi}) = u(0, \hat{\varphi}) \in P_n^0(\hat{\omega})$. The proof of the theorem is complete.

Consider the monodromy operator L of system (0.1), which is defined on the complex extension $\tilde{P}_n^0(\hat{\omega})$ of the space $P_n^0(\hat{\omega})$ by the formula [3] $Lu = X_0(\omega_1, \hat{\varphi})u(\hat{\varphi} - \hat{e}\omega_1)$. Obviously, this operator is linear, continuous, and continuously invertible. [One has $L^{-1}u = X_0^{-1}(\omega_1, \hat{\varphi} + \hat{e}\omega_1)u(\hat{\varphi} + \hat{e}\omega_1)$, the continuity of the matrix function X_0^{-1} follows from the continuity and invertibility of the matrix function X_0.]

The following assertion was proved in [6].

Lemma 1.1. *If a number λ_0 is a regular point of the operator L, then each number λ such that $|\lambda| = |\lambda_0|$ is the regular point of L.*

Lemma 1.2. *The following relations hold for each $p \in Z$.*
(a) $L^p u = X_0(p\omega_1, \hat{\varphi})u(\hat{\varphi} - \hat{e}p\omega_1)$.
(b) $X_0(\varphi_1 + p\omega_1, \hat{\varphi}) = X_0(\varphi)X_0(p\omega_1, \hat{\varphi} - \hat{e}\varphi_1)$, $\varphi_1 \in R$, $\hat{\varphi} \in R^{m-1}$.

Proof. Let $p \in N$; then, by induction, from item (d) of Lemma 0.1, we obtain the relations

$$\begin{aligned} L^p u &= X_0(\omega_1, \hat{\varphi})X_0(\omega_1, \hat{\varphi} - \hat{e}\omega_1) \cdots X_0(\omega_1, \hat{\varphi} - \hat{e}(p-1)\omega_1)u(\hat{\varphi} - \hat{e}p\omega_1) \\ &= X_0(p\omega_1, \hat{\varphi})u(\hat{\varphi} - \hat{e}p\omega_1) \end{aligned}$$

and

$$\begin{aligned} X_0(\varphi_1 + p\omega_1, \hat{\varphi}) &= X_0(\varphi)X_0(\omega_1, \hat{\varphi} - \hat{e}\varphi_1) \cdots X_0(\omega_1, \hat{\varphi} - \hat{e}(\varphi_1 + (p-1)\omega_1)) \\ &= X_0(\varphi)X_0(p\omega_1, \hat{\varphi} - \hat{e}\varphi_1). \end{aligned}$$

It follows from the first relation that $L^{-p}u = X_0^{-1}(p\omega_1, \hat{\varphi} + \hat{e}p\omega_1)u(\hat{\varphi} + \hat{e}p\omega_1)$. In the second relation, we set $\varphi_1 = -p\omega_1$; then $E = X_0(-p\omega_1 + p\omega_1, \hat{\varphi}) = X_0(-p\omega_1, \hat{\varphi})X_0(p\omega_1, \hat{\varphi} + \hat{e}p\omega_1)$ and $X_0^{-1}(p\omega_1, \hat{\varphi} + \hat{e}p\omega_1) = X_0(-p\omega_1, \hat{\varphi})$. Consequently,

$$L^{-p}u = X_0^{-1}(p\omega_1, \hat{\varphi} + \hat{e}p\omega_1)u(\hat{\varphi} + \hat{e}p\omega_1) = X_0(-p\omega_1, \hat{\varphi})u(\hat{\varphi} + \hat{e}p\omega_1).$$

The proof of relation (a) is complete. Let us complete the proof of relation (b):

$$X_0(\varphi_1 - p\omega_1 + p\omega_1, \hat{\varphi}) = X_0(\varphi_1 - p\omega_1, \hat{\varphi})X_0(p\omega_1, \hat{\varphi} - \hat{e}(\varphi_1 - p\omega_1));$$

then

$$X_0(\varphi_1 - p\omega_1, \hat{\varphi}) = X_0(\varphi)X_0^{-1}(p\omega_1, \hat{\varphi} - \hat{e}(\varphi_1 - p\omega_1)) = X_0(\varphi)X_0(-p\omega_1, \hat{\varphi} - \hat{e}\varphi_1).$$

The proof of the lemma is complete.

Theorem 1.2. *For each function* $F \in P_n^0(\omega)$, *there exists a unique function* $u_0 \in P_n^0(\hat{\omega})$ *such that the function* u *given by* (1.2) *is an element of the space* $P_n^0(\omega)$ *if and only if the spectrum of the operator* L *does not meet the unit circle* $|\lambda| = 1$.

Proof. Necessity. Suppose that, for each function $F \in P_n^0(\omega)$, there exists a function $u_0 \in P_n^0(\hat{\omega})$ described in the theorem; then

$$0 = u(\omega_1, \hat{\varphi}) - u_0(\hat{\varphi}) = X_0(\omega_1, \hat{\varphi})u_0(\hat{\varphi} - \hat{e}\omega_1) + \int_0^{\omega_1} X(\omega_1, \hat{\varphi}; \xi)F(\xi, \hat{\varphi} - \hat{e}\omega_1 + \hat{e}\xi)\, d\xi - u_0(\hat{\varphi}),$$

whence we obtain the relation

$$X_0(\omega_1, \hat{\varphi})u_0(\hat{\varphi} - \hat{e}\omega_1) - u_0(\hat{\varphi}) = -\int_0^{\omega_1} X(\omega_1, \hat{\varphi}; \xi)F(\xi, \hat{\varphi} - \hat{e}\omega_1 + \hat{e}\xi)\, d\xi. \tag{1.3}$$

From item (c) of Lemma 0.1, we have $X(\omega_1, \hat{\varphi}; \xi) = X_0(\omega_1, \hat{\varphi})X_0^{-1}(\xi, \hat{\varphi} - \hat{e}\omega_1 + \hat{e}\xi)$. For F, we take the function

$$F(\varphi) = -\frac{6}{\omega_1^3} X_0(\varphi)X_0^{-1}(\omega_1, \hat{\varphi} - \hat{e}\varphi_1 + \hat{e}\omega_1)h(\hat{\varphi} - \hat{e}\varphi_1 + \hat{e}\omega_1)\varphi_1(\omega_1 - \varphi_1),$$

$\varphi_1 \in [0, \omega_1]$, and we continue it as a periodic function of φ_1 outside the interval $[0, \omega_1]$, where h is an arbitrary function from $P_n^0(\hat{\omega})$; then

$$X_0(\omega_1, \hat{\varphi})u_0(\hat{\varphi} - \hat{e}\omega_1) - u_0(\hat{\varphi}) = h(\hat{\varphi}).$$

It follows from the existence and uniqueness of the solution of this equation for an arbitrary right-hand side that $\lambda = 1$ is a regular point of the operator L. By Lemma 1.1, all points lying on the unit circle are regular.

Sufficiency. Suppose that the spectrum of L does not meet the unit circle. The right-hand side of system (1.3) for each function $F \in P_n^0(\omega)$ is an element of the space $P_n^0(\hat{\omega})$; therefore, this system has a unique solution $u_0 \in P_n^0(\hat{\omega})$. Consider the function u given by (1.2), where u_0 is a solution of system (1.3). This function is periodic in $\hat{\varphi}$ and continuous in $\varphi \in R^m$ by virtue of the periodicity and continuity of the right-hand side of (1.2). Let us show that it is periodic in φ_1. For an arbitrary fixed $\hat{\varphi}$, the functions $u(t, \hat{e}t + \hat{\varphi})$ and $u(t + \omega_1, \hat{e}t + \hat{\varphi})$ satisfy system ($1.1_{\hat{\varphi}}$); moreover, since $u(\omega_1, \hat{\varphi}) = u(0, \hat{\varphi})$, we have $u(t, \hat{e}t + \hat{\varphi}) = u(t + \omega_1, \hat{e}t + \hat{\varphi})$ by virtue of the uniqueness of the solution for all $t \in R$. Hence, it follows that $u(\varphi_1 + \omega_1, \hat{\varphi}) = u(\varphi)$, $\varphi \in R^m$; therefore, $u \in P_n^0(\omega)$. The proof of the theorem is complete.

2. HOMOGENEOUS SYSTEM

In forthcoming considerations, we need the following definition.

Definition 2.1 [4, p. 31; 5, p. 233]. System (0.1) is said to be *exponentially dichotomous* if the space $H = H(t)$ of its solutions can be represented as the direct sum $H_1 \dotplus H_2$ of subspaces $H_1 = H_1(t)$ and $H_2 = H_2(t)$ and there exist positive constants M_1, M_2, γ_1, and γ_2 such that

$$|x(t)| \le M_1 e^{-\gamma_1(t-\tau)}|x(\tau)|, \qquad -\infty < \tau \le t < +\infty, \tag{2.1}$$

for $x \in H_1$ and

$$|x(t)| \ge M_2 e^{\gamma_2(t-\tau)}|x(\tau)|, \qquad -\infty < \tau \le t < +\infty, \tag{2.2}$$

for $x \in H_2$.

Theorem 2.1. *System* (0.1) *is exponentially dichotomous if and only if the spectrum of the operator* L *does not meet the unit circle.*

Proof. Necessity. If system (0.1) is exponentially dichotomous then, for each continuous conditionally periodic function f, system (1.1) has a unique continuous almost periodic solution [5, pp. 274–275]. Since the module of this solution lies in the minimum module containing the union of the spectra of the functions f and a [5, p. 275], it follows that such a solution is conditionally periodic with frequencies $\beta_1, \dots, \beta_m$. Then the spectrum of L does not meet the unit circle by Theorems 1.1 and 1.2.

Sufficiency. If the spectrum σ of L does not meet the unit circle, then $\sigma = \sigma_1 \cup \sigma_2$, where σ_1 is the part of the spectrum lying inside the unit circle and σ_2 is the part lying outside it. The space $P_n^0(\hat{\omega})$ can be represented as the direct sum of invariant subspaces $P_{n1}^0(\hat{\omega})$ and $P_{n2}^0(\hat{\omega})$ of the operator L, $P_{nk}^0(\hat{\omega}) = \Pi_k P_n^0(\hat{\omega})$, where the Π_k are the spectral projections of the operator L corresponding to σ_k, $k = 1, 2$.

Let L_k be the restriction of the operator L to the subspace $P_{nk}^0(\hat{\omega})$; then [5, p. 290] there exist numbers $K > 0$ and $q \in (0, 1)$ such that

$$\|L_1^s\| \le K q^s, \qquad \|L_2^{-s}\| \le K q^s, \qquad s \in N. \tag{2.3}$$

Let $x_0 \in P_{n1}^0(\hat{\omega})$, $t = \xi + \tau$, $\xi = \xi_1 + r\omega_1$, $\tau = \tau_1 + p\omega_1$, $\xi_1, \tau_1 \in [0, \omega_1]$, $r, p \in Z$, and $r \ge 0$. The vector function $x(t, \hat{\varphi}, x_0) = X_0(t, \hat{e}t + \hat{\varphi}) x_0(\hat{\varphi})$ with $\hat{\varphi} = 0$ is a solution of system (0.1) [and, for an arbitrary $\hat{\varphi}$, is a solution of system $(1.1_{\hat{\varphi}})$, where $f = 0$].

To prove inequality (2.1), we use Lemma 1.2, the inequality [4, p. 32] $|x(t, 0, x_0)| \le c_1 |x(\zeta, 0, x_0)|$, which holds for some c_1 and for all t and ζ such that $|t - \zeta| \le 2\omega_1$, and the inequality

$$\|X_0(\xi_1 + \tau_1, \hat{\varphi})\| \le c_2,$$

which holds for some c_2 for any $\hat{\varphi}$ and $|\xi_1 + \tau_1| \le 2\omega_1$; then we obtain

$$\begin{aligned}
|x(t, 0, x_0)| &= |X_0(et) x_0(0)| \le \|X_0(t, \hat{e}t + \hat{\varphi}) x_0(\hat{\varphi})\| \\
&= \|X_0(\xi_1 + \tau_1, \hat{e}t + \hat{\varphi}) X_0((r + p)\omega_1, \hat{e}(r + p)\omega_1 + \hat{\varphi}) x_0(\hat{\varphi})\| \\
&\le c_2 \|X_0((r + p)\omega_1, \hat{e}(r + p)\omega_1 + \hat{\varphi}) x_0(\hat{\varphi})\| = c_2 \|X_0((r + p)\omega_1, \hat{\varphi}) x_0(\hat{\varphi} - \hat{e}(r + p)\omega_1)\| \\
&= c_2 \|L_1^{r+p} x_0\| \le c_2 K q^r \|L_1^p x_0\| = c_2 K e^{-\xi_1 \omega_1^{-1} \ln q} e^{\xi \omega_1^{-1} \ln q} \|X_0(p\omega_1, \hat{e}p\omega_1 + \hat{\varphi}) x_0(\hat{\varphi})\| \\
&\le (c_2 K e^{-\xi_1 \omega_1^{-1} \ln q} \le c_3,\ \gamma_1 = -\omega_1^{-1} \ln q > 0) \le c_3 e^{-\gamma_1 \xi} \|x(p\omega_1, \hat{\varphi}, x_0)\|.
\end{aligned} \tag{2.4}$$

In a similar way, one can prove the inequality

$$|x(\tau, 0, x_0)| \le c_4 e^{-\gamma_2 \xi} \|x((p + r)\omega_1, \hat{\varphi}, x_0)\|, \tag{2.5}$$

where $\gamma_2 = -\omega_1^{-1} \ln q > 0$ and $x_0 \in P_{n2}^0(\hat{\omega})$.

It follows from inequalities (2.4) and (2.5) that the expansion $x_0 = x_{01} + x_{02}$, $x_{0k} = \Pi_k x_0$, of an arbitrary element $x_0 \in P_n^0(\hat{\omega})$ implies the expansion $x_0(0) = x_{01}(0) + x_{02}(0)$ of an arbitrary element $x_0(0) \in R^n$. This expansion is unique. Indeed, if we assume that $\tilde{x}_0 \ne x_0$, $\tilde{x}_0(0) = x_0(0)$ and $\tilde{x}_0 = \tilde{x}_{01} + \tilde{x}_{02}$, $\tilde{x}_{0k} = \Pi_k \tilde{x}_0$, then $x_{01}(0) - \tilde{x}_{01}(0) = \tilde{x}_{02}(0) - x_{02}(0)$, and for $x_{01}(0) - \tilde{x}_{01}(0) \ne 0$ the solution of system (0.1) equal to $x_{01} - \tilde{x}_{01}$ for $t = 0$ is both infinitely large and infinitely small as $t \to +\infty$.

Therefore, the space R^n can be represented as the direct sum of subspaces $H_1(0)$ and $H_2(0)$; moreover, the solutions of system (0.1) issuing for $t = 0$ from the subspace $H_1(0)$ tend to zero, and those issuing from $H_2(0)$ tend to infinity as $t \to +\infty$. Likewise, if, instead of $\hat{\varphi} = 0$, we fix an arbitrary $\hat{\varphi}_0 \in R^{m-1}$, then the expansion

$$x_0(\hat{\varphi}_0) = x_{01}(\hat{\varphi}_0) + x_{02}(\hat{\varphi}_0), \qquad x_{0k}(\hat{\varphi}_0) = (\Pi_k x_0)(\hat{\varphi}_0)$$

is independent of the choice of the function x_0 and depends only on the value taken by this function at the point $\hat{\varphi}_0$. In particular, hence, it follows that if $x_0 \in P_{n1}^0(\hat{\omega})$ and $y \ne x_0$, but $y(\hat{\varphi}_0) = x_0(\hat{\varphi}_0)$, then $(\Pi_1 y)(\hat{\varphi}_0) = y(\hat{\varphi}_0)$.

Let us return to inequality (2.4), where $x_0 \in P^0_{n1}(\hat\omega)$ and $x(p\omega_1, \hat\varphi, x_0) = (L_1^p x_0)(\hat e p\omega_1 + \hat\varphi)$. Set $\tilde x_0 = L_1^{-p}\Pi_1 x_0(p\omega_1, 0, x_0) \in P^0_{n1}(\hat\omega)$. (The argument of the operator Π_1 is a constant function of $\hat\varphi$.) Then

$$x(p\omega_1, \hat\varphi, \tilde{x_0}) = (L_1^p \tilde x_0)(\hat\varphi + p\omega_1\hat e) = (\Pi_1 x(p\omega_1, 0, x_0))(\hat\varphi + p\omega_1 \hat e),$$

and since

$$\begin{aligned} x(p\omega_1, 0, \tilde{x_0}) &= (\Pi_1 x(p\omega_1, 0, x_0))(p\omega_1\hat e) \\ &= (\Pi_1((L_1^p x_0)(p\omega_1\hat e)))(p\omega_1 \hat e) = (L_1^p x_0)(p\omega_1\hat e) = x(p\omega_1, 0, x_0), \end{aligned}$$

it follows from the uniqueness of the solution of system (0.1) that $\tilde x_0(0) = x_0(0)$. Let us replace x_0 in inequality (2.4) by $\tilde x_0$; then

$$\begin{aligned} |x(t,0,x_0)| = |x(t,0,\tilde x_0)| &\le c_3 e^{-\gamma_1\xi}\|x(p\omega_1, \hat\varphi, \tilde x_0)\| = c_3 e^{-\gamma_1 \xi}\|\Pi_1 x(p\omega_1, 0, x_0)\| \\ &\le c_3 e^{-\gamma_1\xi}\|\Pi_1\||x(p\omega_1, 0, x_0)| \le c_3 c_1 e^{-\gamma_1\xi}\|\Pi_1\||x(\tau, 0, x_0)|. \end{aligned}$$

The proof of inequality (2.1) is complete; inequality (2.2) can be proved in a similar way. The proof of the theorem is complete.

Note that the monodromy operator L_{φ_0} of the system

$$\frac{dx}{dt} = A(et + \varphi_0)x \tag{2.6}$$

given by the relation $L_{\varphi_0}u = X(\varphi_{10} + \omega_1, \hat\varphi + \hat\varphi_0; \varphi_{10})u(\hat\varphi - \hat e\omega_1)$ is similar to the operator L for any $\varphi_0 \in R^m$; consequently, their spectra coincide. To prove the desired assertion, consider the shift operator $S_{\varphi_0}u = u(\hat\varphi + \hat\varphi_0)$ and the operator of multiplication by the matrix $\mathcal{X}u = X_0(\varphi_{10}, \hat\varphi)u(\hat\varphi)$, $u \in \tilde P^0_n(\hat\omega)$. It follows from Lemma 0.1 that

$$X(\varphi_{10} + \omega_1, \hat\varphi + \hat\varphi_0; \varphi_{10}) = X_0(\varphi_{10}, \hat\varphi + \hat\varphi_0)X_0(\omega_1, \hat\varphi + \hat\varphi_0 - \hat e\varphi_{10})X_0^{-1}(\varphi_{10}, \hat\varphi + \hat\varphi_0 - \hat e\omega_1);$$

therefore, $L_{\varphi_0} = S_{\hat\varphi_0}\mathcal{X}S^{-1}_{\hat e\varphi_{10}}LS_{\hat e\varphi_{10}}\mathcal{X}^{-1}S^{-1}_{\hat\varphi_0}$. We have thereby proved the following assertion.

Theorem 2.2. *System* (2.6) *is exponentially dichotomous for any* $\varphi_0 \in R^m$ *if and only if the spectrum of the operator* L *does not meet the unit circle.*

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